

Higher Order Perturbations Around Backgrounds with One Non-Homogeneous Dimension

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ABSTRACT: It is shown that perturbations around backgrounds with one non-homogeneous dimension, namely of co-homogeneity 1, can be canonically simplified, a property that is shown to hold to any order in perturbation theory. Recalling that the problem naturally reduces to 1d, a procedure is described whereby for each gauge function in 1d two 1d fields are eliminated from the action – one is gauge and can be eliminated without a constraint and the other is auxiliary. These results generalize the results of hep-th/0609001 from linear to non-linear perturbations and they unify two cases of physical interest: cosmological perturbations and perturbations to static spherically symmetric backgrounds. An application to black strings is discussed in some detail.

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1. Introduction

Analysis of perturbations (gravitational waves) around a given background in General Relativity, and in particular making a judicious choice of gauge, is a problem whose complexity grows with the degree of co-homogeneity of the background, namely the number of non-homogeneous coordinates. For co-homogeneity zero, we are dealing with the well-understood case of a maximally symmetric space time (flat space, de-Sitter or Anti de-Sitter) and once harmonic analysis is employed the problem reduces altogether to algebra. The case of co-homogeneity one was solved at the linear level in [1] following a concrete analysis of the black hole negative mode [2] thereby unifying known results in two cases of physical interest: static spherically symmetric space-times (mostly black holes) [3, 4] and spatially homogeneous cosmologies [5, 6]. The solution for co-homogeneity two and higher is unknown and one may expect it to be challenging given the complexity of a specific case, that of perturbations around the Kerr spacetime (where Newman-Penrose variables are employed).

Let us briefly review the results of [1]. It was shown that once the natural reduction to a 1d theory is carried out, for each 1d gauge function two 1d fields can be eliminated: one through the gauge and the other being auxiliary. The transformation to the new set of fields is local in the non-homogeneous coordinate and invertible. This procedure proves that in 1d the gauge can be completely eliminated and an invertible transformation exists into gauge-invariant fields, which is presumably the reason that the cosmological version of this procedure is known as “gauge invariant perturbation theory”. One should stress that while it is trivially true that all physical observables including perturbations must be gauge invariant it is certainly not true in general that one can transform (one to one and locally) into gauge-invariant fields – this property holds only for co-homogeneity one (or zero). Actually, one can work in the action formulation and perform a *constraint free gauge-fix* of certain fields (the constraints are contained already in the equations of motion). Moreover, not only can the gauge be eliminated thereby eliminating a similar number of fields, but also each 1d gauge function is responsible for a 1d auxiliary field which helps to further decouple the action. Accordingly it is said that “the gauge shoots twice”. This auxiliary field is sometimes considered to be a constraint, though from the current perspective it is

certainly not one: a constraint is a residual equation of motion left after fixing the gauge while here the gauge gets eliminated altogether.

The results of [1] were applied to obtain new insight into gravitational waves in the Schwarzschild background (Regge-Wheeler and Zerilli equations) [7] and to a study of the dependence of the black hole negative mode on the space-time dimension [8].

One wonders whether the features of linear perturbations carry over to the non-linear case and in particular whether each 1d gauge function can be continued to be used to completely eliminate two 1d fields. This question on non-linear perturbations is further motivated by applications including cosmology as well as for the computation of the order of phase transition associated with the Gregory-Laflamme black string instability [9, 10, 11].

Experience suggests that the linear part of the equations sets much of their qualitative properties. Indeed, in section 2 we provide and justify a procedure that answers affirmatively and to all orders the questions from the preceding paragraph. This work can be thought to unify the existing literature on non-linear cosmological perturbations [12, 13, 14, 15] with that of non-linear perturbations in spherical symmetry [16]. In section 3 we spell out in greater detail the application of the general procedure to the non-uniform string, but we leave its implementation to future work.

Comment: This paper is not intended to be submitted to a refereed journal. Papers such as this, which suggest a new computational procedure or theory, are easier to referee once they are applied and demonstrated in full rather than as an abstract argument, since the former confirm the paper's validity and usefulness beyond doubt. Indeed, I plan to apply the procedure presented here to the case of the non-uniform black string. On the other hand, it is plain that a general procedure together with its rationale are self-contained and have a value on their own that merits their publication. Thus in order to avoid unnecessary difficulties in the process of refereeing I choose to post this paper on the archives, but postpone submitting it to a refereed journal until it can be strengthened by an application.

2. The procedure

We start this section by stating a canonical procedure to handle non-linear 1d actions with gauge, and then we continue to explain and justify it. The procedure is

- Identify the fields and their gauging and thereby the derivatively-gauged (DG) fields, namely 1d tensors or vectors.
- It is more economical to fix a gauge before obtaining the action. For each gauge function we may eliminate any single field (in whose variation the gauge function appears), but not a DG field. This is a constraint-free gauge-fixing [1].
- Obtain the action (up to the required order in the fields).
- Write the equations of motion for the DG fields.
- Use the equations of motion to solve for the DG fields. The solution is obtained through series inversion and should be expanded up to the required order.

- Substitute the expression for the DG fields back into the action.

Alternatively, one may skip the gauge-fixing step, and upon substituting the DG fields the action will be found to depend only on certain gauge-invariant fields.

Let us now discuss each item in detail.

Fields and general set-up. After separation of variables, namely expanding all fields in harmonic functions of the homogeneous coordinates, we may reduce the perturbation problem to 1d by performing the integration over the homogeneous coordinates. More generally we consider any 1d action with a gauge symmetry. We denote the 1d variable ¹ by x and the fields and gauge fields respectively by the vector notation

$$\begin{aligned}\phi &= \phi^i(x) & i &= 1, \dots, n_F \\ \xi &= \xi^a(x) & a &= 1, \dots, n_G\end{aligned}\tag{2.1}$$

where n_F, n_G are the number of (real) fields and gauge functions, respectively.

Identifying the derivatively-gauged (DG) fields. Let us recall the definition of the DG fields [1]. We assume the gauging to be linear in the gauge functions and at most linear in derivatives, which is indeed the case when analyzing perturbations in GR or gauge theory. The gauge variation is given by

$$\delta\phi = G_1 \xi' + G_0 \xi\tag{2.2}$$

where $(G_1(x), G_0(x, \phi, \phi'))$ is a pair of $n_F \times n_G$ matrices which depend on the coordinate x . We allow G_0 to depend also on the fields ϕ and their derivatives (non-linear contributions to the gauging), but we assume G_1 to have no such dependence, which is indeed the case in GR perturbations.

Definition: The image of G_1 in field space is called *the derivatively-gauged (DG) fields*.

We recognize this algebraic definition to correspond to the standard gauge variation of fields which are vectors or tensors from the 1d point of view.

Following the identification of the DG fields, the field space can be split into the DG-subspace with coordinates ϕ_{DG}^α and an arbitrarily chosen complement with coordinates ϕ_X^r , namely

$$\phi^i = (\phi_{DG}^\alpha, \phi_X^r) \quad \alpha = 1, \dots, n_G, \quad r = 1, \dots, n_F - n_G.\tag{2.3}$$

Obtaining the action. The action

$$S = S[\phi]\tag{2.4}$$

is considered as the input to the procedure. In practice one needs to reduce the perturbation problem to 1d as already mentioned. To simplify the computation a constraint-free gauge-fix should be performed by fixing for each gauge function any field (in whose variation the gauge function appears), as long as it is not a DG field.

¹In the cosmological set-up x would normally be denoted by t , while in spherically symmetric backgrounds it would normally be denoted by r .

Writing the equations of motion for the DG fields. This is a straightforward variation of the action.

Solving for the DG-fields order by order and substituting back into the action. The equations of motion for the DG fields read

$$0 = \frac{\delta S}{\delta \phi_{DG}} = L_{DG} \phi_{DG} + \mathcal{L}_m(\phi_X) + \mathcal{O}(\phi_X, \phi_{DG})^2 . \quad (2.5)$$

This expression highlights the linear part of the equations as obtained in [1]. We suppressed the indices, L_{DG} is just a x -dependent $n_G \times n_G$ matrix which we assume to be invertible, and \mathcal{L}_m is a linear operator with at most one derivative which mixes ϕ_X into the equations of motion of ϕ_{DG} .

We proceed to perform a change of variables from ϕ_{DG} to $\tilde{\phi}_{DG}$ and substitute it back into the action. However, in practice we shall find that it suffices to substitute zero for $\tilde{\phi}_{DG}$ thereby somewhat simplifying the procedure. Accordingly we start by explaining the change of variables, but for practical purposes one may wish to skip to the paragraph of (2.8).

We define

$$\tilde{\phi}_{DG} := L_{DG}^{-1} \frac{\delta S}{\delta \phi_{DG}} = \phi_{DG} + L_{DG}^{-1} \mathcal{L}_m(\phi_X) + \mathcal{O}(\phi_X, \phi_{DG})^2 \quad (2.6)$$

where we used (2.5) in the second equality to recognize that at the linear level $\tilde{\phi}_{DG}$ is a shifted version of ϕ_{DG} .

In order to substitute the change of variables into the action we need to invert this relation and obtain $\phi_{DG} = \phi_{DG}(\tilde{\phi}_{DG}, \phi_X)$. Since we are dealing with a perturbation theory it suffices to perform the inversion perturbatively, and since L_{DG} is invertible that can always be done through a standard procedure as we shall detail shortly.

Next we substitute the inverted relation into the action. Note that the determinant of the transformation is no longer unity, as was the case for linear perturbations, but there is no need to account for that as long as classical physics is involved. Actually by construction (2.6) the equations of motion state that

$$\tilde{\phi}_{DG} = 0 \quad (2.7)$$

and hence this sector is guaranteed to decouple, and it suffices to substitute $\tilde{\phi}_{DG} \rightarrow 0$. Note that equations (2.7) can be interpreted to define a natural complement in field space to the DG fields, rather than the arbitrarily complement X chosen before. Moreover, since zero is gauge-invariant so are the $\tilde{\phi}_{DG}$ fields.

Having realized that $\tilde{\phi}$ can be put to zero, let us describe how this could be implemented in practice to avoid defining $\tilde{\phi}$ in the first place. We start from the equations of motion (2.5), which we now view as an implicit definition for

$$\phi_{DG} = \phi_{DG}(\phi_X) . \quad (2.8)$$

In the context of perturbation theory it suffices to solve for ϕ_{DG} perturbatively. This can always be done given that L_{DG} is invertible (and local in 1d) as we proceed to detail.

One expands ϕ_{DG} as

$$\phi_{DG} = \phi_{DG}^{(1)}(\phi_X) + \phi_{DG}^{(2)}(\phi_X) + \dots \quad (2.9)$$

where the superscript denotes the order with respect to the fields ϕ_X . Next one substitutes this expansion into the equations (2.5) and solves them order by order. At each order one obtains an equation of the form $0 = L_{DG} \phi_{DG}^{(k)} + Src^{(k)}$ where the k 'th order source term depends only upon ϕ_X as well as lower order of ϕ_{DG} . Now $\phi_{DG}^{(k)}$ can be solved for by the assumed invertibility of L_{DG} .

The last step is to substitute (2.8) into the action.

Altogether we have arrived at the canonical form of the action and completed our description of the process.

3. Application to the non-uniform string

In this section we spell out in greater detail how the general procedure applies to the case of the non-uniform string.

The goals for such a computation include

- The order of the GL phase transition requires expanding the action to fourth order, and was preformed in several works [9, 10, 11] the last one employing the Landau-Ginzburg approach. The current method can be applied to further improve on the method by allowing to use a single “master” field, rather than a collection of three fields.
- Performing computations along the non-uniform string branch to a higher order than previously achieved. Such computations could be compared to the numerical results of [17, 18]. If one is interested only in thermodynamics then it is necessary to compute the action up to sixth order to go beyond the existing literature. On the other hand, if one is interested in the solutions themselves, only part of the third order was computed so far.
- Improving and extending the results on charged non-uniform strings [19, 20]. However, in this section we shall restrict ourselves to the neutral case for concreteness.

Order of GL phase transition

Following is the improved procedure to compute the order of the GL phase transition at various space-time dimensions d .

Set-up and identification of the DG fields. The space time coordinates are the standard t, r, z and Ω_{d-2} . t runs along the Euclidean time dimension which has length β asymptotically, r is the radial coordinate, z runs along the compact dimension whose asymptotic size is L , and Ω_{d-2} represents the angular coordinates on the $d-2$ sphere. The total space-time dimension is $d+1$. The gauge functions obeying the isometries are ξ_r, ξ_z (where ξ_μ is the parameter of an infinitesimal diffeomorphism) while the DG fields are those which are a

vector or tensor from the r coordinate point of view, namely h_{rr} and h_{rz} (where $h_{\mu\nu}$ is the perturbation to the metric).

Ansatz. We need an ansatz which keeps the DG fields and it is convenient to take the following constraint-free gauge choice

$$ds^2 = f(r) dt^2 + f(r)^{-1} e^{2b} dr^2 + e^{2\beta(r)} (dz - \alpha dr)^2 + r^2 e^{2c} d\Omega_{d-2}^2, \quad (3.1)$$

where b, α, c are functions of (r, z) , β is a function of r alone and $f(r) = 1 - (r_0/r)^{d-3}$. The uniform black string is obtained upon setting all the fields to zero. In fixing the gauge we note that for z -dependent modes we have two gauge functions, namely ξ_r, ξ_z which we can use to eliminate two fields chosen here to be h_{tt} and h_{zz} . However, for the zero mode fields the situation is somewhat different [2]. As ξ_z and α do not have a zero mode consistent with the symmetries only ξ_r can be used to eliminate a single field, chosen here to be h_{tt} and hence we retain the zero mode of β (or equivalently h_{zz}).

Compute the $d + 1$ dimensional action. Compute the Gibbons Hawking action. This could be achieved by computing the Einstein-Hilbert action and integrating by parts to avoid terms which include two r -derivatives acting on a single field. For the purpose of computing the order of the transition it is enough to retain terms up to fourth order in the fields according to the Landau-Ginzburg theory.

Reduction of the action to 1d. Only the z integration is non-trivial and we need to expand the fields in harmonic functions. The action has two symmetries which constrain the expansion. The first is $z \rightarrow -z$, and the second is $z \rightarrow z + L/2$ together with $\epsilon \rightarrow -\epsilon$, where ϵ is the small expansion parameter along the non-uniform branch. Both symmetries are remnants of the z -translation symmetry of the uniform string. Expand

$$\begin{aligned} b(r, z) &= \epsilon b_1 \cos(kz) + \epsilon^2 b_0 + \epsilon^2 b_2 \cos(2kz) \\ \alpha(r, z) &= \epsilon \alpha_1 \sin(kz) + \epsilon^2 \alpha_2 \sin(2kz) \end{aligned} \quad (3.2)$$

and c is expanded just like b . All fields are now functions of r alone. Expand the action up to the fourth order in ϵ and perform all z integrations.

Integrate out the DG fields. The DG fields are the harmonics of b, α namely the five 1d fields $b_1, b_0, b_2, \alpha_1, \alpha_2$. For the purpose of this application it is not required to perturbatively solve an implicit equation. Next substitute the solutions for the DG fields back into the action. Finally, terms with a single derivative can be eliminated in some circumstances through integration by parts.

A Test. The expressions for the quadratic action should agree with those derived in [2],

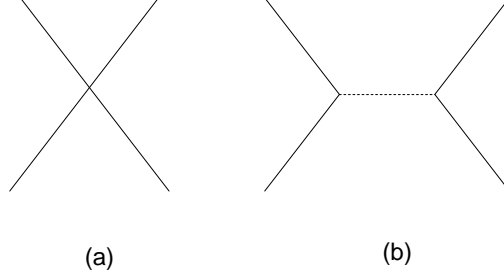


Figure 1: The expansion of the action up to fourth order in terms of Feynman diagrams. The solid four external legs represent the c_1 field and the dashed internal propagator in (b) represents all possible fields, namely c_0, c_2, β . (a) is the quartic vertex of the GL mode while (b) describes the contribution from the back-reaction fields.

namely

$$\begin{aligned}
S_2 &= S_2[c_1] + S_2[c_2] + S_2[\beta] + S_2[c_0] \\
S_2[c_1] &= \frac{(d-1)(d-2)}{2} L \int dt r^{d-2} dr [f c_1'^2 - k^2 c_1^2 + V(r) c_1^2] \\
V(r) &= -\frac{2(d-1)(d-3)^3}{r^2 (2(d-2)r^{d-3} - (d-1))^2} \\
S_2[\beta] &= \frac{d-1}{d-2} L \int dt f^2 r^{d-2} dr \beta'^2.
\end{aligned} \tag{3.3}$$

The expression for $S_2[c_2]$ is gotten from that of $S_2[c_1]$ by replacing $k \rightarrow 2k$ while the expression for $S_2[c_0]$ can be deduced from eq. (4.13) in [2].

First order. Write the equation of motion for c_1 and solve the corresponding eigenvalue equation to obtain $c_1 \equiv \phi_{GL}(r)$ and $k_0^2 \equiv \lambda_{GPY}$.

Second order. Solve the back-reaction equations of motion for the fields c_2, c_0, h .

Substitution in the action. Compute $S_4(GL)$ and $S_2(BR)$ - see [11] for the definitions of these notations. Continue to compute $\mathcal{C} = S_4(GL) - S_2(BR)$ which determines the order of the phase transition in the canonical ensemble (see also [19]). This computation is encoded by the Feynman diagrams of figure 1. The results can be compared against table 2 of [11]. It is possible to compute also the coefficient \mathcal{A} , allowing full thermodynamic information including to pass to the micro-canonical ensemble as in [11].

Higher orders

The more efficient expressions gotten here should facilitate the computation of higher orders. The perturbation theory can be conveniently organized by means of Feynman diagrams. For instance, in order to expand the action (or equivalently the free energy and obtain the thermodynamics) up to order j we need to compute all tree diagrams (since it is a classical field theory) with j external c_1 legs, and substitute $c_1 \equiv \phi_{GL}(r)$. In this way we compute the free energy $F = F(\epsilon)$. Canonically the free energy is a function of temperature, or equivalently of k in our case. To obtain $F(k)$ we should compute also $k = k(\epsilon) = k^{(0)} + \epsilon^2 k^{(2)} + \dots$ to the required order.

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